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Stability of the minimal heterotic standard model bundle

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Abstract: The observable sector of the "minimal heterotic standard model" has precisely the matter spectrum of the MSSM: three families of quarks and leptons, each with a right-handed neutrino, and one Higgs–Higgs conjugate pair. In this paper, it is explicitly proven that the $SU(4)$ holomorphic vector bundle leading to the MSSM spectrum in the observable sector is slope-stable.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua.

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JHEP06(2006)032

Contents

JHEP06(2006)032

1. Introduction

The $E_8 \times E_8$ heterotic string [1−3] is, perhaps, the simplest context in which to construct string compactifications giving rise to a realistic matter spectrum; that is, three families of quarks/leptons and one (perhaps several) Higgs–Higgs conjugate pairs without any exotic representations or any other vector-like pairs. Within the last year, there has been significant progress in building such models [4−15]. In the vacua presented in [4−6], called heterotic standard models, the observable sector has the MSSM matter spectrum with the addition of one extra pair of Higgs fields. In [7] the number of Higgs pairs was reduced to one, yielding the exact MSSM matter spectrum in the observable sector. Hence, the vacua in [7] are called "minimal" heterotic standard models. The MSSM matter spectrum has been obtained, in different contexts, in [8, 9].

In this paper, we will confine our discussion to the heterotic standard model vacua presented in [4 – 7]. Their basic construction is as follows. As is well known, the whole matter content of the standard model, including the right-handed neutrino [16 – 18] fits into the **16** and **10** representations of $Spin(10)$. To embed this unification of quarks/leptons into the $E_8 \times E_8$ heterotic string, one has to break the observable sector E_8 gauge group appropriately. This can be done by choosing a suitable gauge instanton [19 – 26] as the vacuum field configuration on a Calabi-Yau threefold. In particular, an $SU(4)$ instanton leaves a $Spin(10)$ gauge group unbroken [4]. The corresponding rank 4 vector bundle is constructed via the method of bundle extensions [27 – 29]. Of course, the $Spin(10)$ gauge group must be further broken to a group containing the standard model gauge group as a factor. The obvious mechanism is to add Wilson lines [30 – 33], thus breaking $Spin(10)$ directly at the compactification scale. In particular, we use a $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson line to break down to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$. In order to do so, the Calabi-Yau threefold must have a large enough fundamental group [22, 34 – 36], that is, it must contain a $\mathbb{Z}_3 \times \mathbb{Z}_3$. A Calabi-Yau threefold whose fundamental group is exactly $\mathbb{Z}_3 \times \mathbb{Z}_3$ was constructed in [37], and is used in [4 – 7]. The low energy particle spectrum can then be computed using methods of algebraic geometry as discussed in [38 – 40].

An important phenomenological aspect of heterotic standard model vacua is the $U(1)_{B-L}$ factor occurring in the low energy gauge group. Usual nucleon decay is suppressed in [4 – 7] by a large compactification mass of $O\big(10^{16}\big)$ GeV. In addition, these theories exhibit natural doublet-triplet splitting, thus suppressing proton decay via dimension five operators. The role of the gauged $U(1)_{B-L}$ symmetry is to disallow any $\Delta L = 1$ and $\Delta B = 1$ dimension four terms that would lead to the disastrous decay of nucleons [41]. Of course, this symmetry must be spontaneously broken at the order of the electroweak scale. This will be discussed elsewhere [42]. Hence, only the usual Yukawa couplings and a possible Higgs μ-term can occur in the superpotential at the renormalizable level. Geometrically, these couplings are cubic products of cohomology groups and restricted by classical geometry. The effect of the elliptic fibration of the Calabi-Yau threefold on the Yukawa texture was analyzed in [43], and leads to one naturally light quark/lepton family.

An essential requirement of these vacua is that the holomorphic vector bundle used in the observable sector be slope-stable. This guarantees [44, 45] that the associated gauge connection satisfies the hermitian Yang-Mills equations and, hence, preserves $\mathcal{N} = 1$ supersymmetry. The vector bundles in the observable sector of [4 – 6] were shown to be slope-stable in [46]. In this paper, we present an analogous proof that the $SU(4)$ vector bundle in the minimal heterotic standard model [7] is, indeed, slope-stable as well. Thus, the observable sector containing exactly the matter spectrum of the MSSM is $\mathcal{N} = 1$ supersymmetric.

The structure of the hidden sector is less clear. There is a maximal dimensional subcone (codimension zero) of the Kähler cone where the observable sector bundle is slope-stable and the hidden sector satisfies the Bogomolov bound. Hence, there is no obstruction to constructing anomaly free vacua whose hidden sector bundle is slope-stable. However, we have not explicitly constructed such a hidden sector bundle. Nor is it entirely clear that this is desirable. As discussed in [47 – 49], the necessity to stabilize all moduli at a point with

JHEP06(2006)032

a small positive cosmological constant [50] might require that the vacuum, in the heterotic case, contain anti-five-branes. If the moduli can be stabilized for such a configuration then, for example, a trivial hidden sector bundle (which is trivially slope-stable) can be chosen. This issue will be discussed in detail elsewhere [51]. We note that the slope-stability of both the observable and hidden sector bundles was proven for the vacuum in [9].

2. The Calabi-Yau manifold

2.1 Double fibration

Let us start by describing the underlying Calabi-Yau threefold. We begin with an elliptic fibration over a rational elliptic (dP_9) surface. Such an elliptic fibration is automatically a fiber product

$$\widetilde{X} \stackrel{\text{def}}{=} B_1 \times_{\mathbb{P}^1} B_2 \tag{2.1}$$

of two dP_9 surfaces B_1 and B_2. In the following, we always choose surfaces with suitable $\mathbb{Z}_3 \times \mathbb{Z}_3$ automorphisms [37] yielding a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on $\widetilde{X}$.

Although not strictly necessary, let us discuss the geometry of the $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant divisors. The class f of the elliptic fiber on each dP_9 surfaces B_i, $i = 1, 2$, is of course $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant. There is one [6] additional invariant divisor class, which we call t. It is a three-section of the elliptic fibration $B_i \to \mathbb{P}^1$. These $b_2(B_i)^{\mathbb{Z}_3\times\mathbb{Z}_3} = 2$ invariant divisor classes on B_i give rise to $b_3(\widetilde{X})^{\mathbb{Z}_3\times\mathbb{Z}_3} = 3$ invariant divisor classes on the fiber product as follows. Two of them are

$$\tau_1 \stackrel{\text{def}}{=} \pi_1^{-1}(t)\,, \quad \tau_2 \stackrel{\text{def}}{=} \pi_2^{-1}(t)\,, \tag{2.2}$$

three-sections of the elliptic fibrations π_2 and π_1, respectively. The third invariant divisor is the Abelian surface (T^4)

$$\phi \stackrel{\text{def}}{=} \pi_1^{-1}(f) = \pi_2^{-1}(f)\,. \tag{2.3}$$

There is a commutative square of projections

$$\begin{tikzcd}
\dim_{\mathbb{C}} = 3: & & \left(\widetilde{X},\, K_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}\right) \arrow[dl, "\pi_1", "K_{\widetilde{X}|B_1} = \mathcal{O}_{\widetilde{X}}(\phi)"'] \arrow[dr, "\pi_2"', "K_{\widetilde{X}|B_2} = \chi_1^2\mathcal{O}_{\widetilde{X}}(\phi)"] & \\
\dim_{\mathbb{C}} = 2: & \left(B_1,\, K_{B_1} = \mathcal{O}_{B_1}(-f)\right) \arrow[dr, "\beta_1", "K_{B_1|\mathbb{P}^1} = \chi_1^2\mathcal{O}_{B_1}(f)"'] & & \left(B_2,\, K_{B_2} = \chi_1\mathcal{O}_{B_2}(-f)\right) \arrow[dl, "\beta_2"', "K_{B_2|\mathbb{P}^1} = \mathcal{O}_{B_2}(f)"] \\
\dim_{\mathbb{C}} = 1: & & \left(\mathbb{P}^1,\, K_{\mathbb{P}^1} = \chi_1\mathcal{O}_{\mathbb{P}^1}(-2)\right), &
\end{tikzcd} \tag{2.4}$$

where χ_1, χ_2 are characters [6] of $\mathbb{Z}_3 \times \mathbb{Z}_3$ encoding the equivariant action on bundles.

The quotient

$$X \stackrel{\text{def}}{=} \widetilde{X} \Big/ (\mathbb{Z}_3 \times \mathbb{Z}_3) \tag{2.5}$$

JHEP06(2006)032

is a torus-fibered Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$, which we take to be the base manifold of our string compactification. However, in practice we work with equivariant constructions on the universal cover $\widetilde{X}$. For a free group action these descriptions are equivalent.

2.2 Topology

It is important to understand the even cohomology groups $H^{\text{ev}}(X, \mathbb{Z})$, because that is where the Chern classes live. Rationally, it is clear that

$$H^{\text{ev}}\Big(\widetilde{X}, \mathbb{Q}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = H^{\text{ev}}\Big(X, \mathbb{Q}\Big)\,. \tag{2.6}$$

The degree 2 invariant integral cohomology of $\widetilde{X}$ is

$$H^2\Big(\widetilde{X}, \mathbb{Z}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \operatorname{span}_{\mathbb{Z}}\big\{\tau_1, \tau_2, \phi\big\}\,. \tag{2.7}$$

We can compare it with the cohomology of X using the quotient map

$$q:\ \widetilde{X} \to X \qquad \Rightarrow \qquad q^*:\ H^*(X, \mathbb{Z}) \to H^*(\widetilde{X}, \mathbb{Z})\,. \tag{2.8}$$

In degree 2, the image is an index 3 sub-lattice of $H^2(\widetilde{X}, \mathbb{Z}) \simeq \mathbb{Z}^3$ generated by $\tau_1 - \tau_2$, $3\tau_1$, ϕ. In other words, the equivariant line bundles on $\widetilde{X}$ are of the form

$$\mathcal{O}_{\widetilde{X}}(x_1\tau_1 + x_2\tau_2 + x_3\phi) \qquad x_1, x_2, x_3 \in \mathbb{Z}\,,\ \ x_1 + x_2 \equiv 0 \mod 3\,. \tag{2.9}$$

The products of the degree 2 generators can easily be determined, and one finds relations

$$H^{\text{ev}}\Big(\widetilde{X}, \mathbb{Q}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbb{Q}\big[\tau_1, \tau_2, \phi\big] \Big/ \big\langle \phi^2,\ \tau_i\phi = 3\tau_i^2 \big\rangle\,. \tag{2.10}$$

Hence, every even degree cohomology class can be written as a polynomial in τ_1, τ_2, and ϕ subject to the relations $\phi^2 = 0$ and $\tau_i\phi = 3\tau_i^2$.

3. Visible bundle

3.1 Construction of the bundle

Having presented the Calabi-Yau manifold, we proceed to define a holomorphic rank 4 vector bundle on it. First, define equivariant rank 2 vector bundles

$$\mathcal{V}_1 = \mathcal{O}_{\widetilde{X}}\big(-\tau_1 + \tau_2\big) \otimes \pi_1^*\big(\mathcal{W}_1\big) \tag{3.1a}$$
$$\mathcal{V}_2 = \mathcal{O}_{\widetilde{X}}\big(+\tau_1 - \tau_2\big) \otimes \pi_2^*\big(\mathcal{W}_2\big)\,, \tag{3.1b}$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are rank 2 vector bundles on B_1 and B_2 which we will define in detail in section 6, eqns. (6.1a) and (6.1b). Using these, we define the desired rank 4 vector bundle $\widetilde{\mathcal{V}}$ as an extension

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \widetilde{\mathcal{V}} \longrightarrow \mathcal{V}_2 \longrightarrow 0\,. \tag{3.2}$$

JHEP06(2006)032

Using the fact that the first Chern class of $\mathcal{W}_i$ is trivial, $\wedge^2 \mathcal{W}_i = \mathcal{O}_{B_i}$, we first remark that

$$c_1\big(\widetilde{\mathcal{V}}\big) = 0 \ \in H^2\Big(\widetilde{X}, \mathbb{Z}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \simeq \mathbb{Z}^3 \,. \tag{3.3}$$

But we really want an $SU(4)$ bundle on the quotient $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, that is

$$c_1\Big(\widetilde{\mathcal{V}}\Big/(\mathbb{Z}_3 \times \mathbb{Z}_3)\Big) = 0 \ \in H^2\Big(X, \mathbb{Z}\Big) \simeq \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \,. \tag{3.4}$$

The vanishing of the first Chern class including the torsion part follows from $\wedge^4 \widetilde{\mathcal{V}} = \mathcal{O}_{\widetilde{X}}$, where $\mathcal{O}_{\widetilde{X}}$ stands for the trivial line bundle with the trivial $\mathbb{Z}_3 \times \mathbb{Z}_3$ equivariant group action.

3.2 Non-Trivial Extensions

We defined the rank 4 bundle $\widetilde{\mathcal{V}}$ as a generic extension of the form eq. (3.2). Clearly, we have to make sure that a non-trivial extension exists, since the *trivial extension* $\mathcal{V}_1 \oplus \mathcal{V}_2$ cannot give rise to an irreducible $SU(4)$ instanton. The space of extensions is

$$\begin{aligned}
\mathrm{Ext}^1\Big(\mathcal{V}_2, \mathcal{V}_1\Big) &= H^1\Big(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2^\vee\Big) \\
&= H^1\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-2\tau_1 + 2\tau_2) \otimes \pi_1^*(\mathcal{W}_1) \otimes \pi_2^*(\mathcal{W}_2{}^\vee)\Big) \\
&= H^1\Big(\widetilde{X}, \pi_1^*\big(\mathcal{W}_1 \otimes \mathcal{O}_{B_1}(-2t)\big) \otimes \pi_2^*\big(\mathcal{W}_2 \otimes \mathcal{O}_{B_2}(2t)\big)\Big) \,.
\end{aligned} \tag{3.5}$$

This cohomology group can directly be computed using the Leray spectral sequence and the push-down eqns. (6.20) and (6.21). One obtains

$$H^i\Big(\widetilde{X},\ \mathcal{V}_1 \otimes \mathcal{V}_2^\vee\Big) = \begin{cases} 0 & i = 3, \\ 8R[\mathbb{Z}_3 \times \mathbb{Z}_3] & i = 2, \\ 4R[\mathbb{Z}_3 \times \mathbb{Z}_3] & i = 1, \\ 0 & i = 0. \end{cases} \tag{3.6}$$

where $R[\mathbb{Z}_3 \times \mathbb{Z}_3]$ stands for the regular representation, that is, the sum of all 9 irreducible representations of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Of course, only invariant extensions give rise to equivariant vector bundles $\widetilde{\mathcal{V}}$. The invariant subspace is

$$\mathrm{Ext}^1\Big(\mathcal{V}_2, \mathcal{V}_1\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = H^1\Big(\widetilde{X},\ \mathcal{V}_1 \otimes \mathcal{V}_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = 4 \tag{3.7}$$

is indeed non-zero, so suitable extensions do exist.

3.3 Low-Energy Spectrum

The low energy particle spectrum is determined through the cohomology of $\widetilde{\mathcal{V}}$ and $\wedge^2\widetilde{\mathcal{V}}$ according to the decomposition

$$\mathbf{248} = \big(\mathbf{1}, \mathbf{45}\big) \oplus \big(\mathbf{4}, \mathbf{16}\big) \oplus \big(\overline{\mathbf{4}}, \overline{\mathbf{16}}\big) \oplus \big(\mathbf{6}, \mathbf{10}\big) \oplus \big(\mathbf{15}, \mathbf{1}\big) \tag{3.8}$$

JHEP06(2006)032

under $E_8 \supset SU(4) \times Spin(10)$. It is easy to show that $H^i\big(\widetilde{X}, \widetilde{\mathcal{V}}\big) = 0$ for $i = 0, 2, 3$. Hence a simple index computation yields

$$H^i\Big(\widetilde{X},\ \widetilde{\mathcal{V}}\Big) = \begin{cases} 0 & i = 3, \\ 0 & i = 2, \\ 3R[\mathbb{Z}_3 \times \mathbb{Z}_3] & i = 1, \\ 0 & i = 0. \end{cases} \tag{3.9}$$

Furthermore, interrelated long exact sequences [6] together with

$$H^*\Big(\widetilde{X},\ \wedge^2 \mathcal{V}_1\Big) = H^*\Big(\widetilde{X},\ \wedge^2 \mathcal{V}_2\Big) = 0 \tag{3.10}$$

yield

$$H^i\Big(\widetilde{X},\ \wedge^2 \widetilde{\mathcal{V}}\Big) = H^i\Big(\widetilde{X},\ \mathcal{V}_1 \otimes \mathcal{V}_2\Big) = H^i\Big(\widetilde{X},\ \pi_1^*(\mathcal{W}_1) \otimes \pi_1^*(\mathcal{W}_1)\Big)\,. \tag{3.11}$$

The latter is easily computed using the push-down formula eqns. (6.17) and (6.18) and the Leray spectral sequence. The result is that

$$H^i\Big(\widetilde{X},\ \wedge^2 \widetilde{\mathcal{V}}\Big) = H^i\Big(\widetilde{X},\ \mathcal{V}_1 \otimes \mathcal{V}_2\Big) = \begin{cases} 0 & i = 3, \\ \chi_2 \oplus \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2 & i = 2, \\ \chi_2 \oplus \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2 & i = 1, \\ 0 & i = 0. \end{cases} \tag{3.12}$$

Finally, the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on the cohomology is tensored with the Wilson line, and every state that is not invariant under the combined action is projected out. The regular representations in eq. (3.9) yield 3 full generations of quarks and leptons, each with a right-handed neutrino. More interesting is the Wilson line action on the **10** of $Spin(10)$, which potentially could lead to exotic color triplets ("triplet Higgs"). We chose the Wilson line such that

$$\mathbf{10} = \Big[\chi_2^2\big(\mathbf{1}, \mathbf{2}, 3, 0\big) \oplus \chi_1^2\chi_2^2\big(\mathbf{3}, \mathbf{1}, -2, -2\big)\Big] \oplus \Big[\chi_2\big(\mathbf{1}, \overline{\mathbf{2}}, -3, 0\big) \oplus \chi_1\chi_2\big(\overline{\mathbf{3}}, \mathbf{1}, 2, 2\big)\Big] \tag{3.13}$$

under the decomposition

$$Spin(10) \supset SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times \mathbb{Z}_3 \times \mathbb{Z}_3\,. \tag{3.14}$$

Combining eqns. (3.13) and (3.12), we see that one vector-like pair of Higgs survives the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient while all color triplets are projected out.

4. Slope-stability

4.1 Conditions for stability

We now proceed and show that the Kähler class $\omega \in H^2\big(\widetilde{X}, \mathbb{R}\big)$ can be chosen such that the visible sector vector bundle $\widetilde{\mathcal{V}}$, eq. (3.2), is equivariantly[1] slope-stable. That means

[1] $\widetilde{\mathcal{V}}$ being equivariantly stable is the same as $\widetilde{\mathcal{V}}/\big(\mathbb{Z}_3 \times \mathbb{Z}_3\big)$ being stable. For the remainder of this section, everything is equivariant.

that for all reflexive sub-sheaves $\mathcal{F} \hookrightarrow \widetilde{\mathcal{V}}$, the slope

$$\mu(\mathcal{F}) \stackrel{\text{def}}{=} \frac{1}{\operatorname{rank}\mathcal{F}} \int_{\widetilde{X}} c_1(\mathcal{F}) \wedge \omega^2 \tag{4.1}$$

is negative,

$$\mu(\mathcal{F}) < \mu(\widetilde{\mathcal{V}}) = 0 \tag{4.2}$$

The easiest way to prove this is to derive a set of sufficient inequalities for the Kähler class ω, and then to find a common solution [46]. We note that they are not always necessary, that is, the inequalities are not sharp.

For example, consider only $\mathcal{V}_1$ defined by eqns. (3.1a), (6.1a). Let $\mathcal{L}$ be any sub-line bundle, that is

$$\begin{tikzcd} 0 \arrow[r] & \chi_1 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2-\phi) \arrow[r, "u"] & \mathcal{V}_1 \arrow[r, "v"] & \chi_1^2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2+\phi) \otimes \pi_1^* I_3 \arrow[r] & 0 \\ & & \mathcal{L} \arrow[u, "i"] \arrow[ul, dashed, "w"] \arrow[ur, "v\circ i"'] & & \end{tikzcd} \tag{4.3}$$

The composition $v \circ i$ either vanishes or not. We distinguish the two cases:

$v \circ i = 0$: There exists a non-zero map

$$w: \ \mathcal{L} \to \chi_1 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2-\phi) \tag{4.4}$$

such that $i = u \circ w$.

$v \circ i \neq 0$: There exists a non-zero map

$$v \circ i: \ \mathcal{L} \to \chi_1^2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2+\phi) \tag{4.5}$$

whose image vanishes at the codimension two locus where $\pi_1^* I_3$ vanishes.

The existence of these maps restricts the line bundle $\mathcal{L}$. Now if $\widetilde{\mathcal{V}}$ is stable, then all these line bundles $\mathcal{L}$ must be of negative slope, $\mu(\mathcal{L}) < 0$. We only have to check this inequality for the $\mathcal{L}$ of largest slope, and these form a finite set (see Appendix A):

$v \circ i = 0$:

$$\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2-\phi)\,. \tag{4.6}$$

$v \circ i \neq 0$: The composition $v \circ i$ cannot be an isomorphism, since that would split the short exact sequence eq. (4.3). Hence, $\mathcal{L}$ can only be a proper sub-line bundle, and those of largest slope are

$$\begin{aligned} \Big\{ & \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2),\, \mathcal{O}_{\widetilde{X}}(-4\tau_1+\tau_2+2\phi),\, \mathcal{O}_{\widetilde{X}}(-3\tau_1+\phi), \\ & \mathcal{O}_{\widetilde{X}}(-2\tau_1-\tau_2+\phi),\, \mathcal{O}_{\widetilde{X}}(-\tau_1-2\tau_2+2\phi) \Big\}\,. \end{aligned} \tag{4.7}$$

JHEP06(2006)032

The first line bundle $\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2)$ actually has the same fiber degrees (coefficients of τ_1 and τ_2) as the range of $v \circ i$. Because of the push-down formula eq. (6.12), the largest such sub-line bundle whose image vanishes at $\pi_1^* I_3$ is actually

$$\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2+\phi) \otimes \pi_1^* \circ \beta_1^* \Big(\mathcal{O}_{\mathbb{P}^1}(-3)\Big) = \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2-2\phi)\,. \tag{4.8}$$

Therefore, the possible line bundles $\mathcal{L}$ of largest slope are

$$\begin{aligned}\mathcal{L} \in \Big\{ & \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2-2\phi),\, \mathcal{O}_{\widetilde{X}}(-4\tau_1+\tau_2+2\phi),\, \mathcal{O}_{\widetilde{X}}(-3\tau_1+\phi),\\ & \mathcal{O}_{\widetilde{X}}(-2\tau_1-\tau_2+\phi),\, \mathcal{O}_{\widetilde{X}}(-\tau_1-2\tau_2+2\phi)\Big\}\,.\end{aligned} \tag{4.9}$$

Similarly, one obtains a finite set of potentially destabilizing sub-line bundles of $\mathcal{V}_2$.

Now to prove [46] stability of $\widetilde{\mathcal{V}}$, it suffices to show that

- Sub-line bundles of $\widetilde{\mathcal{V}}$ have negative slope.
- Rank 2 sub-bundles have negative slope. A sufficient criterion is that $\wedge^2\mathcal{V}_1$ has negative slope and that proper sub-line bundles of $\wedge^2\mathcal{V}_2$ are of negative slope.
- Rank 3 sub-bundles (reflexive sheaves) have negative slope $\Leftrightarrow$ sub-line bundles of $\widetilde{\mathcal{V}}^\vee$ have negative slope.

This gives a finite set of line bundles which have to have negative slope. One obtains

Proposition 1. *If all line bundles $\mathcal{O}_{\widetilde{X}}(a_1\tau_1+a_2\tau_2+b\phi)$ with*

$$\begin{aligned}(a_1,a_2,b) \in \Big\{ & (-1,-2,2),\, (2,-2,-1),\, (2,-5,1),\, (-4,1,2),\, (-1,1,-1),\\ & (-2,2,0),\, (-2,-1,2),\, (1,-4,2),\, (1,-1,-1)\Big\}\end{aligned} \tag{4.10}$$

have negative slope, then the vector bundle $\widetilde{\mathcal{V}}$, eq. (3.2)*, is equivariantly stable.*

4.2 Kähler Cone Substructure

The Kähler cone, that is the set of possible Kähler classes, is [46]

$$\mathcal{K} \stackrel{\text{def}}{=} \Big\{ x_1\tau_1+x_2\tau_2+y\phi \,\Big|\, x_1,x_2,y>0 \Big\} \subset H^2\big(\widetilde{X},\mathbb{R}\big) = \langle \tau_1,\tau_2,\phi\rangle_{\mathbb{R}}\,. \tag{4.11}$$

The slope eq. (4.1) of a line bundle obviously depends quadratically on the Kähler parameters x_1, x_2, y, and can be computed [46] to be

$$\mu\Big(\mathcal{O}_{\widetilde{X}}(a_1\tau_1+a_2\tau_2+b\phi)\Big) = 3(x_1x_2+6y)(a_1x_2+a_2x_1)+x_1x_2(3a_1+3a_2+18b)\,. \tag{4.12}$$

Therefore, according to Proposition 1 the vector bundle $\widetilde{\mathcal{V}}$ is stable if the inequalities

$$\begin{array}{lll}
\mu\big(\mathcal{O}_{\widetilde{X}}(-\tau_1-2\tau_2+2\phi)\big) = & 18x_1x_2-6x_1^2-3x_2^2-18yx_2-36yx_1 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(2\tau_1-2\tau_2-\phi)\big) = & -6x_1^2+6x_2^2+36yx_2-36yx_1-18x_1x_2 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(2\tau_1-5\tau_2+\phi)\big) = & -15x_1^2+6x_2^2+36yx_2-90yx_1 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(-4\tau_1+\tau_2+2\phi)\big) = & 18x_1x_2+3x_1^2-12x_2^2-72yx_2+18yx_1 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2-\phi)\big) = & 3x_1^2-3x_2^2-18yx_2+18yx_1-18x_1x_2 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(-2\tau_1+2\tau_2)\big) = & 6x_1^2-6x_2^2-36yx_2+36yx_1 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(-2\tau_1-\tau_2+2\phi)\big) = & 18x_1x_2-3x_1^2-6x_2^2-36yx_2-18yx_1 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(\tau_1-4\tau_2+2\phi)\big) = & 18x_1x_2-12x_1^2+3x_2^2+18yx_2-72yx_1 & <0 \\
\mu\big(\mathcal{O}_{\widetilde{X}}(\tau_1-\tau_2-\phi)\big) = & -3x_1^2+3x_2^2+18yx_2-18yx_1-18x_1x_2 & <0
\end{array} \tag{4.13}$$

are simultaneously satisfied.

It is easy to see that there are many solutions. For example, the Kähler class

$$\omega = 3\Big(2\tau_1+3\tau_2+\phi\Big) \in H^2\Big(\widetilde{X},\mathbb{R}\Big) \tag{4.14}$$

satisfies all the inequalities eq. (4.13), the slopes being -621, -378, -702, -1512, -1269, -594, -918, -27, and -675, respectively. The overall factor of 3 in eq. (4.14) is not essential, but included to make it a $\mathbb{Z}_3\times\mathbb{Z}_3$-equivariant integral cohomology class. In other words, the class is actually primitive in the integral cohomology of the quotient $X=\widetilde{X}/(\mathbb{Z}_3\times\mathbb{Z}_3)$. Of course, in string theory the Kähler form is not quantized. As usual, the radial part of the Kähler class, that is, the overall volume, does not matter for the stability of vector bundles. We conclude from eq. (4.14) that the set

$$\mathcal{K}^s \subset \mathcal{K} \subset H^2\big(\widetilde{X},\mathbb{R}\big) \tag{4.15}$$

of Kähler classes that make all slopes of the line bundles in Proposition 1 negative is not empty. Therefore, the solution set $\mathcal{K}^s$ of the strict inequalities eq. (4.13) must be a maximal-dimensional subcone of the Kähler cone $\mathcal{K}$. Note that all cones have their tip at the origin $0\in H^2\big(\widetilde{X},\mathbb{R}\big)\simeq\mathbb{R}^3$. Hence, we can draw a 2-dimensional "star map" of these cones as they are seen by an observer at the origin. This is depicted in figure 1. One observes that the boundary of the set $\mathcal{K}^s$ is roughly triangular. On the right hand side in figure 1, it is bounded by two curved but smooth faces. Those bounds are an artifact of our proof, and are merely sufficient but not necessary conditions. Although it is in general difficult to determine the precise subcone of the Kähler cone where $\widetilde{\mathcal{V}}$ is stable, one expects it to extend even further to the right. On the other hand, the flat face of $\mathcal{K}^s$ at the left in figure 1 is a boundary saturating a necessary *and* sufficient inequality. It is precisely the locus where the slope of $\mathcal{V}_1$ changes sign, and if one crosses this line then $\mu\big(\mathcal{V}_1\big)>0$ becomes a destabilizing sub-bundle of $\widetilde{\mathcal{V}}$, see eq. (3.2). The interpretation is analogous to the picture of D-branes as complexes; this boundary of $\mathcal{K}^s$ is a line of marginal stability. To its right, the bound state $\widetilde{\mathcal{V}}$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ is stable. To its left, the reversed bound state

$$0\longrightarrow\mathcal{V}_2\longrightarrow\widetilde{\mathcal{V}}_{\text{rev}}\longrightarrow\mathcal{V}_1\longrightarrow 0\,. \tag{4.16}$$

is stable. Using the same methods as above, it is easy to see that $\widetilde{\mathcal{V}}_{\text{rev}}$ is indeed stable in a subcone of $\mathcal{K}$ extending to the left of $\mathcal{K}^s$. Although reversing the short exact sequence

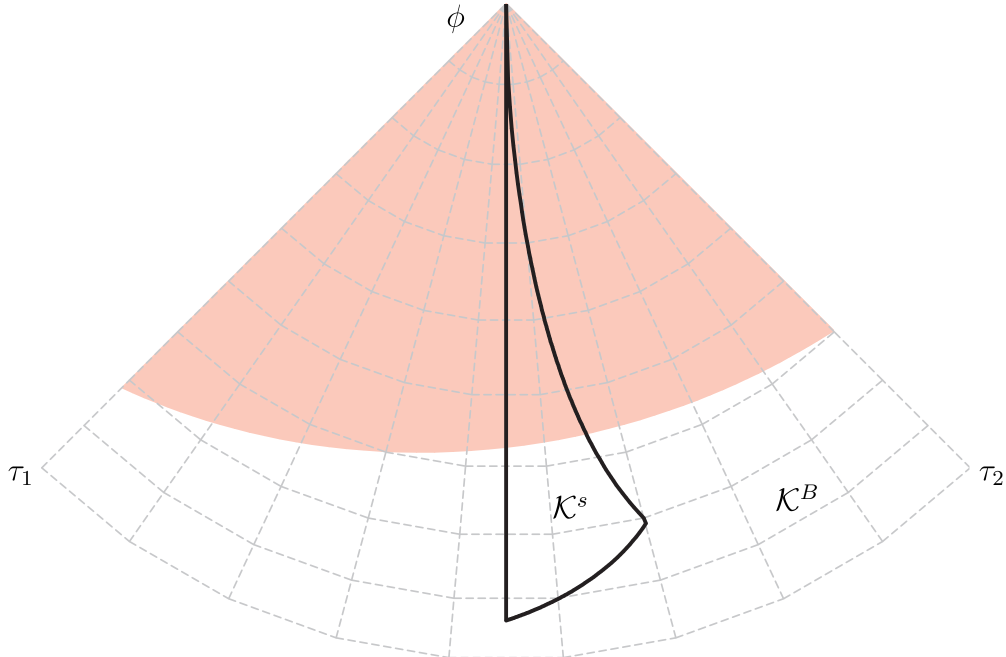

Figure 1: Map projection of the unit sphere intersecting the Kähler cone, that is, the positive octant in $H^2(\widetilde{X}, \mathbb{R}) \simeq \mathbb{R}^3$. The rank 4 bundle $\widetilde{\mathcal{V}}$ is stable inside the black triangular region $\mathcal{K}^s$. In the white region $\mathcal{K}^B$ the Bogomolov inequality allows an $\mathcal{N}=1$ hidden sector, see section 5.

potentially alters the cohomology groups, it turns out that $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{V}}_{\text{rev}}$ give rise to the same low energy spectrum.

To summarize, the observable sector vector bundle $\widetilde{\mathcal{V}}$ is slope-stable with respect to any Kähler class ω in a 3-dimensional subcone $\mathcal{K}^s$ of the 3-dimensional Kähler cone $\mathcal{K}$. The region $\mathcal{K}^s$ is shown explicitly in figure 1. By working harder to strengthen Proposition 1 or by making small changes to the vector bundle it will be possible to enlarge that fraction of the Kähler cone.

5. Hidden sector

Although not the main topic of this paper, in this section we will briefly discuss the hidden sector. Denote by $\widetilde{\mathcal{V}}'$ the holomorphic vector bundle of the hidden sector. For simplicity, we will assume that $c_1(\widetilde{\mathcal{V}}') = 0$, that is, the hidden sector contains an $SU(n)$ gauge instanton. Given the tangent bundle $T\widetilde{X}$ of the Calabi-Yau threefold and the observable sector bundle $\widetilde{\mathcal{V}}$, anomaly cancellation imposes the constraint

$$c_2(\widetilde{\mathcal{V}}') = c_2(T\widetilde{X}) - c_2(\widetilde{\mathcal{V}}) - [C_5]\,. \tag{5.1}$$

Here, $[C_5]$ is the curve class on which five-branes are wrapped. For simplicity, let us assume that $[C_5] = 0$ (both weakly and strongly coupled heterotic string). Then, using

$$c_2(T\widetilde{X}) = 12(\tau_1^2 + \tau_2^2)\,, \qquad c_2(\widetilde{\mathcal{V}}) = \tau_1^2 + 4\tau_2^2 + 4\tau_1\tau_2\,, \tag{5.2}$$

it follows that

$$c_2\big(\widetilde{\mathcal{V}}'\big) = 11\tau_1^2 + 8\tau_2^2 - 4\tau_1\tau_2 = \Big(3\tau_1^2\Big) + 4\Big(\tau_1^2 + \tau_2^2\Big) - 4\Big(\tau_1\tau_2 - \tau_1^2 - \tau_2^2\Big)\,. \tag{5.3}$$

Note that $c_2\big(\widetilde{\mathcal{V}}'\big)$ is neither effective nor antieffective, the terms in brackets being pull-backs of effective curves on X. If $\widetilde{\mathcal{V}}'$ is a slope-stable vector bundle with respect to a Kähler class ω, then it must satisfy the Bogomolov inequality [52]

$$\int_{\widetilde{X}} c_2\big(\widetilde{\mathcal{V}}'\big) \wedge \omega > 0\,. \tag{5.4}$$

Using the parametrization of ω in eq. (4.11), we see that

$$\begin{aligned}\int_{\widetilde{X}} c_2\big(\widetilde{\mathcal{V}}'\big) \wedge \omega &= \int_{\widetilde{X}} \Big(11\tau_1^2 + 8\tau_2^2 - 4\tau_1\tau_2\Big) \wedge \Big(x_1\tau_1 + x_2\tau_2 + y\phi\Big) \\ &= \int_{\widetilde{X}} \Big(4x_1 + 7x_2 - 12y\Big)\tau_1^2\tau_2 = 3\Big(4x_1 + 7x_2 - 12y\Big)\,.\end{aligned} \tag{5.5}$$

Therefore, the Bogomolov inequality is satisfied for any Kähler class for which

$$4x_1 + 7x_2 - 12y > 0\,. \tag{5.6}$$

This defines a 3-dimensional cone in the Kähler moduli space which we denote by $\mathcal{K}^B$. The subcone $\mathcal{K}^B$ is shown as the white region in figure 1. Its complement, where eq. (5.6) is violated, is drawn in pink. Note that the Kähler class eq. (4.14) for which the observable sector vector bundle was proven to be stable also satisfies eq. (5.6). Hence,

$$\mathcal{K}^s \cap \mathcal{K}^B \neq \emptyset\,. \tag{5.7}$$

Since both $\mathcal{K}^s$ and $\mathcal{K}^B$ are open (solutions of strict inequalities), their non-empty intersection is automatically a maximal-dimensional subcone of the Kähler cone. It follows that both $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{V}}'$ can, in principle, be slope-stable with respect to a Kähler class in $\mathcal{K}^s \cap \mathcal{K}^B$. Often, the Bogomolov inequality is the only obstruction to finding stable bundles. However, we have not explicitly constructed such a hidden sector bundle.

6. Serre construction

6.1 General construction

In this section, we are going to construct two $SU(2)$ vector bundles $\mathcal{W}_1$ and $\mathcal{W}_2$ on the dP_9 surfaces B_1 and B_2, respectively. They are defined as extensions of the form

$$0 \longrightarrow \chi_1\mathcal{O}_{B_1}(-f) \longrightarrow \mathcal{W}_1 \longrightarrow \chi_1^2\mathcal{O}_{B_1}(f) \otimes I_3 \longrightarrow 0 \tag{6.1a}$$

$$0 \longrightarrow \chi_2^2\mathcal{O}_{B_2}(-f) \longrightarrow \mathcal{W}_2 \longrightarrow \chi_2\mathcal{O}_{B_2}(f) \otimes I_6 \longrightarrow 0 \tag{6.1b}$$

with the ideal sheaves I_3 and I_6 defined in subsection 6.2. If they satisfy the Cayley-Bacharach property, then $\mathcal{W}_1$ and $\mathcal{W}_2$ are rank 2 vector bundles for generic extensions. We check this in subsection 6.3.

Note that the determinant line bundles are trivial by construction, that is

$$\wedge^2 \mathcal{W}_1 = \mathcal{O}_{B_1}\,, \qquad \wedge^2 \mathcal{W}_2 = \mathcal{O}_{B_2}\,. \tag{6.2}$$

Therefore, the bundles are self-dual,

$$\big(\mathcal{W}_1\big)^\vee = \mathcal{W}_1\,, \qquad \big(\mathcal{W}_2\big)^\vee = \mathcal{W}_2\,. \tag{6.3}$$

6.2 Ideal Sheaves

Let p_1, p_2, p_3 be the singular points of the $3I_1$ Kodaira fibers in $B_1 \to \mathbb{P}^1$. Similarly, let q_1, q_2, q_3 be the singular points of the $3I_1$ Kodaira fibers in $B_2 \to \mathbb{P}^1$. Recall that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is generated by g_1 and g_2, where g_1 acts on the base $\mathbb{P}^1$ and g_2 does not (it is a translation along the elliptic fiber). The $\mathbb{Z}_3 \times \mathbb{Z}_3$ characters are defined via

$$\begin{aligned} \chi_1(g_1) &= \omega & \qquad \chi_1(g_2) &= 1 \\ \chi_2(g_1) &= 1 & \qquad \chi_2(g_2) &= \omega\,, \end{aligned} \tag{6.4}$$

Note that the points p_i and q_j are g_2-fixed points, and that g_2 acts as $\chi_2 \oplus \chi_2^2$ on the tangent spaces $T_{p_i}B_1$ and $T_{q_j}B_2$. First, we define the ideal sheaf I_3 as

$$0 \longrightarrow I_3 \longrightarrow \mathcal{O}_{B_1} \longrightarrow \bigoplus_{i=1,2,3} \mathcal{O}_{p_i} \longrightarrow 0\,. \tag{6.5}$$

Furthermore, define for any $G_2 \simeq \mathbb{Z}_3$ fixed point p the subscheme $Z(p)$ as the point p and its first derivative in χ_2^2-direction. In local coordinates $(x, y) \in \mathbb{C}^2$, this $\mathbb{Z}_3$ group acts as

$$g_2(x,y) = \Big(\chi_2(g_2)\,x, \chi_2^2(g_2)\,y\Big) = \big(\omega x, \omega^2 y\big)\,, \quad \omega \stackrel{\text{def}}{=} e^{\frac{2\pi i}{3}} \tag{6.6}$$

and the scheme $Z(p)$ is

$$Z(p) = \operatorname{spec}\Big(\mathbb{C}[x,y] \big/ \big\langle x, y^2 \big\rangle\Big)\,. \tag{6.7}$$

Define the ideal sheaf I_6 as the sheaf of functions vanishing at $\mathbb{Z}(q_1)$, $Z(q_2)$, and $Z(q_3)$. That is,

$$0 \longrightarrow I_6 \longrightarrow \mathcal{O}_{B_2} \longrightarrow \bigoplus_{i=1,2,3} \mathcal{O}_{Z(q_i)} \longrightarrow 0\,. \tag{6.8}$$

In other words, I_6 are the functions vanishing at q_i and whose first derivative in the χ_2^2 direction vanishes.

6.3 Cayley-Bacharach Property

Recall the Cayley-Bacharach property for an extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{W} \longrightarrow \mathcal{M} \otimes I_n \longrightarrow 0 \tag{6.9}$$

of line bundles $\mathcal{L}$, $\mathcal{M}$ and ideal sheaf I_n of n points on a surface B. It has the Cayley-Bacharach property if the sections

$$s \;\in H^0\Big(B,\; \mathcal{L}^\vee \otimes \mathcal{M} \otimes K_B\Big) \tag{6.10}$$

vanishing at $n-1$ points of the ideal sheaf automatically vanish at the n-th point. The Cayley-Bacharach property implies that $\mathcal{W}$ is generically a rank 2 vector bundle.

First, let us check that $\mathcal{W}_1$, eq. (6.1a), has the Cayley-Bacharach property. The sections in question are

$$
\begin{aligned}
s_1 \;&\in H^0\Big(B_1,\ \mathcal{O}_{B_1}(-f)^\vee \otimes \mathcal{O}_{B_1}(f) \otimes K_{B_1}\Big) \\
&= H^0\Big(B_1,\ \mathcal{O}_{B_1}(f)\Big) = H^0\Big(\mathbb{P}^1,\ \mathcal{O}_{\mathbb{P}^1}(1)\Big) .
\end{aligned}
\tag{6.11}
$$

Furthermore, the ideal sheaf I_3 vanishes at 3 points in 3 different fibers. But a section of $\mathcal{O}_{B_1}(f)$ can only vanish at one fiber, or it is identically zero. Hence, a section s_1 vanishing at 2 of the 3 points vanishes automatically at the 3-rd, and the Cayley-Bacharach property holds. The extension $\mathcal{W}_2$, eq. (6.1b), satisfies Cayley-Bacharach analogously. Therefore, $\mathcal{W}_1$ and $\mathcal{W}_2$ are rank 2 vector bundles.

6.4 Push-Down Formulae

To compute the cohomology groups of vector bundles, we always utilize the Leray spectral sequence. For that, we need to know the push-down of all bundles involved.

First, consider the ideal sheaves. A standard application of the long exact sequence for the push-down to eq. (6.5) immediately yields

$$
\beta_{1*}\big(I_3\big) = \mathcal{O}_{\mathbb{P}^1}(-3) , \qquad R^1\beta_{1*}\big(I_3\big) = \chi_1 \mathcal{O}_{\mathbb{P}^1}(-1) . \tag{6.12}
$$

For the push-down of I_6 defined in eq. (6.8), first note that according to the definition of $Z(q_i)$ the push-down of the skyscraper sheaves are

$$
\beta_{2*}\mathcal{O}_{Z(q_i)} = \mathcal{O}_{\beta_{2*}(q_i)} \oplus \chi_2^2 \mathcal{O}_{\beta_{2*}(q_i)} . \tag{6.13}
$$

The long exact sequence for the push-down contains a non-zero coboundary map which can be computed as in [6]. One finds that

$$
\beta_{2*}\big(I_6\big) = \mathcal{O}_{\mathbb{P}^1}(-3) , \qquad R^1\beta_{2*}\big(I_6\big) = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \left[\bigoplus_{i=1}^{3} \chi_2^2 \mathcal{O}_{\beta_2(q_i)}\right] . \tag{6.14}
$$

Using the push-down of the ideal sheaves, we find the long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \chi_1 \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & \beta_{1*}\big(\mathcal{W}_1\big) \arrow[r] & \chi_1^2 \mathcal{O}_{\mathbb{P}^1}(-2) \arrow[dll, "\delta"] \\
& \chi_1^2 \mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] & R^1\beta_{1*}\big(\mathcal{W}_1\big) \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] & 0 .
\end{tikzcd}
\tag{6.15}
$$

From the discussion is subsection 6.3 we know that $\mathcal{W}_1 = {\mathcal{W}_1}^\vee$ is a vector bundle, that is, it satisfies the relative duality for vector bundles

$$
R^1\beta_{1*}\big(\mathcal{W}_1\big) = \Big(\beta_{1*}\big(\mathcal{W}_1\big) \otimes K_{B_1|\mathbb{P}^1}\Big)^\vee . \tag{6.16}
$$

This uniquely fixes the coboundary map δ to be an isomorphism, and one obtains

$$
\begin{aligned}
\beta_{1*}\mathcal{W}_1 &= \chi_1 \mathcal{O}_{\mathbb{P}^1}(-1) , \\
R^1\beta_{1*}\mathcal{W}_1 &= \mathcal{O}_{\mathbb{P}^1} .
\end{aligned}
\tag{6.17}
$$

JHEP06(2006)032

The coboundary map in the analogous push-down of $\mathcal{W}_2$ is zero for trivial reasons. We find that

$$\begin{aligned} \beta_{2*}\mathcal{W}_2 &= \chi_2^2\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \chi_2\mathcal{O}_{\mathbb{P}^1}(-2)\,, \\ R^1\beta_{2*}\mathcal{W}_2 &= \chi_2^2\mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_2\mathcal{O}_{\mathbb{P}^1}\,. \end{aligned} \tag{6.18}$$

Finally, we need the push-down of $\mathcal{W}_i \otimes \mathcal{O}_{B_i}(2t)$. These are simpler to compute since the fiber degrees are large, so $R^1\beta_{i*}$ vanishes. First, the push-down of the ideal sheaves twisted by $\mathcal{O}_{B_i}(2t)$ is

$$\beta_{1*}\Big(I_3 \otimes \mathcal{O}_{B_1}(2t)\Big) = 3\mathcal{O}_{\mathbb{P}^1} \oplus 3\mathcal{O}_{\mathbb{P}^1}(-1)\,, \qquad R^1\beta_{1*}\Big(I_3 \otimes \mathcal{O}_{B_1}(2t)\Big) = 0\,, \tag{6.19a}$$

$$\beta_{2*}\Big(I_6 \otimes \mathcal{O}_{B_2}(2t)\Big) = 6\mathcal{O}_{\mathbb{P}^1}(-1)\,, \qquad R^1\beta_{2*}\Big(I_6 \otimes \mathcal{O}_{B_2}(2t)\Big) = 0\,. \tag{6.19b}$$

The push-down long exact sequence for $\mathcal{W}_1$, $\mathcal{W}_2$ splits [6], and we obtain

$$\begin{aligned} \beta_{1*}\Big(\mathcal{W}_1 \otimes \mathcal{O}_{B_1}(2t)\Big) &= 6\mathcal{O}_{\mathbb{P}^1}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^1} \oplus 3\mathcal{O}_{\mathbb{P}^1}(1)\,, \\ R^1\beta_{1*}\Big(\mathcal{W}_1 \otimes \mathcal{O}_{B_1}(2t)\Big) &= 0\,, \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} \beta_{2*}\Big(\mathcal{W}_2 \otimes \mathcal{O}_{B_2}(2t)\Big) &= 6\mathcal{O}_{\mathbb{P}^1}(-1) \oplus 6\mathcal{O}_{\mathbb{P}^1}\,, \\ R^1\beta_{2*}\Big(\mathcal{W}_2 \otimes \mathcal{O}_{B_2}(2t)\Big) &= 0\,. \end{aligned} \tag{6.21}$$

The push-down for $\mathcal{W}_i \otimes \mathcal{O}_{B_i}(-2t)$ can be obtained by relative duality.

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A. Line bundles

By elementary computation of $\mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\widetilde{X}})$, one can easily see that every equivariant sub-line bundle $\mathcal{L}$ of $\mathcal{O}_{\widetilde{X}}$ is

$$\mathcal{O}_{\widetilde{X}}(-\phi),\ \mathcal{O}_{\widetilde{X}}(-3\tau_1+\phi),\ \mathcal{O}_{\widetilde{X}}(-2\tau_1-\tau_2),\ \mathcal{O}_{\widetilde{X}}(-\tau_1-2\tau_2),\ \mathcal{O}_{\widetilde{X}}(-3\tau_2+\phi) \tag{A.1}$$

or a sub-line bundle thereof. Since a sub-line bundle of a line bundle always has smaller slope, the equivariant sub-line bundles of $\mathcal{O}_{\widetilde{X}}$ of largest slope are those listed in eq. (A.1).

JHEP06(2006)032

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